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STABILITY AND WELL-POSEDNESS IN VECTOR LEXICOGRAPHIC EQUILIBRIUM PROBLEMS

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ABSTRACT

The vector equilibrium problems have numerous applications in mathematical physics, game theory, transportation, mechanics, economics, and includes optimization, fixed-point problems and variational inequalities. Inspired by the great importance of equilibrium problems and the lexicographic order, we consider vector equilibrium problems using the lexicographic order. Using classes of generalized continuous function, we establish sufficient conditions for the stability of solution including closedness, semicontinuity, continuity properties of solution mappings. Sufficient conditions for a family of such problems to be (uniquely) well-posed at the reference point are established. Inasmuch as the equilibrium problems contains many problems related to optimizations, our results can be applied to derive the corresponding results for such special cases,…

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1 INTRODUCTION

The equilibrium problems were first introduced by Blum and Oettli (1994). These problems have been playing an important role in optimization theory with many outstanding applications particularly in transportation, mechanics, economics, etc. The mathematical formulations of equilibrium problems incorporate many other important problems related to optimization, namely, optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers (see, e.g., Flores, 2011; Bianchi *et al*., 2005; Djafari *et al*., 2005; Hai and Khanh, 2007; Sadeqi and Alizadeh, 2011).

A relatively new but rapidly growing topic is the stability and sensitivity analysis of solutions, including semicontinuity properties in the sense of Berge and Hausdorff (see, e.g., Anh and Khanh, 2010, 2004; Bianchi and Pini, 2006), the Hölder/Lipschitz continuity of solution mappings (see, e.g., Ait Mansour and Riahi, 2005; Anh and Khanh, 2008, 2007; Bianchi and Pini, 2006, 2003; Li *et al*., 2009; Li and Li, 2011), and the (unique) well-posedness of approximate solutions in the sense of Hadamard and Tikhonov (see, e.g., Anh *et al*., 2009; Anh and Khanh, 2014, 2012, 2011; Fang *et al*., 2008; Morgan and Scalzo, 2006; Noor and Noor, 2005). The ultimate issue of computational methods for solving equilibrium problems has also been considered in the literature, see, e.g., Burachik and Kassay, 2012; Iusem and Sosa, 2003; Muu and Oettli, 1992 and the references therein.

With regard to vector equilibrium problems, most of existing results in the literature correspond to the case when the order is induced by a closed convex cone in a vector space. Therefore, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization, see, e.g., Bianchi *et al.*, 2010, 2007; Carlson, 2010; Emelichev *et al.*, 2010; Freuder *et al.,* 2010; Konnov, 2003; Küçük *et al.,* 2011; Mäkelä *et al.,* 2012. However, for equilibrium problems, most of papers have been focused on the issue of solvability/existence. We observed the papers Anh *et al.,* 2015, 2014, which devoted to continuity and well-posedness for lexicographic vector equilibrium problems in Banach spaces. Of course, such important topics as stability and well-posedness must be the aims of many works, including stability and well-posedness for the problems related to optimization.

In this article, we study necessary and/or sufficient conditions for such problems in metric spaces to be stable and well-posed. To simplify the presentation, most of the results are formulated for the case when the objective function takes its values in $\mathbb{R}^2$. The general, n-dimensional, cases are not significantly different.

The rest of the paper is organized as follows: Section 2 is devoted to problem statements and preliminary facts. In Section 3, we study the sufficient conditions for the solution mappings of considered problems to be closed, upper semicontinuous, and continuous. Section 4 is focused on well-posedness for lexicographic vector equilibrium problems. Concluding remarks in Section 5 summarize the main results and propose possible developments.

2 PRELIMINARIES

Throughout the paper, if not otherwise specified, $\mathbb{R}^n$ denotes an n-dimensional Banach space, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For a subset A of a topological space, int$A$ and cl$A$ stand for the interior and closure, respectively (resp.), of A. We first recall the concept of lexicographic cone in finite dimensional spaces and models of equilibrium problems with the order induced by a such cone.

The lexicographic cone of $\mathbb{R}^n$, denoted C_l, is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, i.e.,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n \mid \exists\, i \in \{1,2,\dots,n\}: x_i > 0, \text{and } x_j = 0, \forall j < i\}.$$

By the definition of the lexicographic cone C_l, it is not hard to see that this cone is convex and pointed, and induces the total order as follow:

$$x \geq_l y \Leftrightarrow x - y \in C_l.$$

Moreover, we also observe that it is neither closed nor open.

In what follows, $K: \Lambda \rightrightarrows X$ is a set-valued mapping between metric spaces and $f = (f_1, f_2, \dots, f_n): K(\Lambda) \times K(\Lambda) \times \Lambda \longrightarrow \mathbb{R}^n$ is a vector-valued function. For each $\lambda \in \Lambda$, we consider the following lexicographic vector equilibrium problem:

(LEP$_\lambda$) Find $\bar{x} \in K(\lambda)$ such that

$$f(\bar{x}, y, \lambda) \geq_l 0, \forall y \in K(\lambda).$$

In the sequel, we restrict ourselves to the case $n = 2$, since the general case is similar, thereby (LEP$_\lambda$) can be equivalently stated as follows:

(LEP$_\lambda$) Find $\bar{x} \in K(\lambda)$ such that

$$f_1(\bar{x}, y, \lambda) \geq 0, \quad \forall y \in K(\lambda),$$
$$f_2(\bar{x}, z, \lambda) \geq 0, \quad \forall z \in Z(\bar{x}, \lambda).$$

Here, the set-valued mapping $Z: K(\Lambda) \times \Lambda \rightrightarrows X$ is defined by

$$Z(x,\lambda) := \begin{cases} \{z \in K(\lambda) \mid f_1(x,z,\lambda) = 0\}, & \text{if } (\lambda, x) \in \text{graph } S_1 \\ X & \text{otherwise.} \end{cases}$$

where $S_1: \Lambda \rightrightarrows X$ denotes the solution mapping of the scalar equilibrium problem determined by the real-valued function f_1:

$$S_1(\lambda) := \{x \in K(\lambda) \mid f_1(x, y, \lambda) \geq 0, \forall y \in K(\lambda)\}.$$

Recall that graph Q stands for the graph of a (set-valued) mapping $Q: \Lambda \rightrightarrows X$:

$$\text{graph } Q := \{(\lambda, x) \in \Lambda \times X \mid x \in Q(\lambda)\}.$$

It is worth noting that this model covers bilevel optimization problems: minimize $g_2(\cdot, \lambda)$ over the solution set of the problem of minimizing $g_1(\cdot, \lambda)$ over $K(\lambda)$, where g_1 and g_2 are real-valued functions on graph K.

We denote (**LEP**) $:= \{(\text{LEP}_\lambda) \mid \lambda \in \Lambda\}$ with the solution mapping $S: \Lambda \rightrightarrows X$. Since the existence conditions have been intensively studied, in this paper we only focuss on stability and well-posedness for the problems and always assume that the solution sets are nonempty at the considered point.

We first recall some notions.

Definition 2.1 (e.g., Aubin and Frankowska, 1990). Let $Q: X \rightrightarrows Y$ be a set-valued mapping between metric spaces.

(i) Q is said to be *upper semicontinuous* (*usc*) at $x_0 \in \text{domQ} := \{x \in X \mid Q(x) \neq \emptyset\}$ if, for any open subset U of Y with $Q(x_0) \subseteq U$, there is a neighborhood N of x_0 such that $Q(x) \subseteq U$ for all $x \in N$.

(ii) Q is said to be *lower semicontinuity* (*lsc*) at $x_0 \in \text{domQ}$ if, for each open subset U of Y with $Q(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that $Q(x) \cap U \neq \emptyset$ for all $x \in N$

Q is said to be continuous at x_0 if it is both usc and lsc at x_0.

(iii) Q is said to be *closed* at $x_0 \in \text{domQ}$ if, from $(x_n, y_n) \in \text{graph}Q$ tending to (x_0, y_0), it follows that $(x_0, y_0) \in \text{graph}Q$.

We say that Q has a property in $A \subseteq X$ if Q has it at any point in A. Of course, in this case "at x_0" is deleted. We will often use the following well-known facts (e.g., see Anh and Khanh, 2009).

(a) Q is usc at x_0 if and only if, for each superset U of $Q(x_0)$, and for each sequence $x_n \to x_0$ in X, there is n_0 such that for all $n \geq n_0$, $Q(x_n) \subset U$.

(b) Q is lsc at x_0 if and only if, for all $x_n \to x_0$ and $y \in Q(x_0)$, there exists $y_n \in Q(x_n)$ such that $y_n \to y$.

The following relaxed continuity properties of functions are also needed.

Definition 2.2 (Morgan and Scalzo, 2004). Let X be a metric space and $g: X \to \overline{\mathbb{R}}$

(i) g is said to be *upper* (*lower*, respectively) *semicontinuous*, written shortly as usc (lsc, resp), at x_0 if, for all sequences $\{x_n\}$ convergent to x_0, $g(x_0) \geq \limsup g(x_n)$ ($g(x_0) \leq \liminf g(x_n)$, resp).

(ii) g is said to be *upper pseudocontinuous* at $x_0 \in X$ if

$$[g(x) > g(x_0)] \Longrightarrow [g(x) > \limsup g(x_n), \forall x_n \to x_0].$$

(iii) g is said to be *lower pseudocontinuous* at $x_0 \in X$ if

$$[g(x) < g(x_0)] \Longrightarrow [g(x) < \liminf g(x_n), \forall x_n \to x_0].$$

(iv) g is said to be *pseudocontinuous* at $x_0 \in X$ if it is both lower and upper pseudocontinuous at this point.

Of course, upper semicontinuity (lower semicontinuity) implies upper pseudocontinuity (lower pseudocontinuity, resp). The class of the pseudocontinuous functions properly contains that of the continuous functions as shown by the following.

Example 2.1. The function $g: \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x+1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ x-1, & \text{if } x < 0 \end{cases}$$

is pseudocontinuous, but neither upper nor lower semicontinuous at 0.

Definition 2.3 (Bianchi and Pini, 2003). Let $g: X \times X \longrightarrow \overline{\mathbb{R}}$.

(i) g is said to be *pseudomonotone* if, for all $x, y \in X$, $g(x,y) \geq 0 \Longrightarrow g(y,x) \leq 0$;

(ii) g is said to be *quasimonotone* if, for any $x, y \in X$, $g(x,y) > 0 \Longrightarrow g(y,x) \leq 0$.

Lemma 2.1 (Morgan and Scalzo, 2006). A function $g: X \to \overline{\mathbb{R}}$ is pseudocontinuous in X if and only if, for all $\{x_n\}$ and $\{y_n\}$ in X converging to x and y, resp,

$$[g(y) < g(x)] \Longrightarrow [\limsup g(y_n) < \liminf g(x_n)].$$

3 STABILITY OF SOLUTION SETS OF (LEP)

In this section we discuss stability property of solution sets of (LEP), such as upper semicontinuity, lower semicontinuity of the solution mappings of (LEP).

Theorem 3.1. *For* (LEP$_\lambda$), *assume that*

(i) K is continuous at $\bar{\lambda}$ and $K(\bar{\lambda})$ is compact;

(ii) f_i is upper semicontinuous in $K(\bar{\lambda}) \times K(\bar{\lambda}) \times \{\bar{\lambda}\}$ for $i = 1,2$;

(iii) Z is lsc in $S_1(\bar{\lambda}) \times \{\bar{\lambda}\}$.

Then, the solution map S is both usc and closed at $\bar{\lambda}$.

Proof. (a) First, we show that S_1 is usc and closed at $\bar{\lambda}$. Suppose, to derive a contradiction, that there exists an open superset U of $S_1(\bar{\lambda})$ such that there are sequences $\{\lambda_n\} \to \bar{\lambda}$ and $x_n \in S_1(\lambda_n) \setminus U$ for all n. Combining the upper semicontinuity of K with the compactness of $K(\bar{\lambda})$, we can assume that there is $x_0 \in K(\bar{\lambda})$ such that $\{x_n\} \to x_0$ (taking a subsequence if necessary).

If $x_0 \notin S_1(\bar{\lambda})$, there exists $y_0 \in K(\bar{\lambda})$ such that

$$f_1(x_0, y_0, \bar{\lambda}) < 0.$$

The lower semicontinuity of K at $\bar{\lambda}$ derives the existence of $y_n \in K(\lambda_n)$, $\{y_n\} \to y_0$. As $x_n \in S_1(\lambda_n)$, we imply that

$$f_1(x_n, y_n, \lambda_n) \geq 0.$$

Thanks to (ii), $f_1(x_0, y_0, \bar{\lambda}) \geq 0$, a contradiction. Therefore, $x_0 \in S_1(\bar{\lambda}) \subseteq U$, which is a contradiction with the fact that $x_n \notin U$ for all n. Thus, S_1 is usc at $\bar{\lambda}$.

Next, for any $\{\lambda_n\} \to \bar{\lambda}$ and $\{x_n\} \subseteq S_1(\lambda_n)$ with $\{x_n\} \to x_0$. Using the given argument as above, we imply that $x_0 \in S_1(\bar{\lambda})$, i.e., S_1 is closed at $\bar{\lambda}$.

(b) Finally, we prove that S is upper semicontinuous at $\bar{\lambda}$. Arguing by contradiction suppose that there are an open set $U \supseteq S(\bar{\lambda})$, $\{\lambda_n\} \to \bar{\lambda}$, such that $x_n \in S(\lambda_n) \setminus U$ for all n. Since S_1 upper semicontinuity of at $\bar{\lambda}$ and $S_1(\bar{\lambda})$ is compact, $\{x_n\} \to x_0$ for (a subsequence and) some $x_0 \in S_1(\bar{\lambda})$.

If $x_0 \notin S(\bar{\lambda})$, there is $y_0 \in Z(x_0, \bar{\lambda})$ such that $f_2(x_0, y_0, \bar{\lambda}) < 0$. The lower semicontinuity of Z in turn yields $y_n \in Z(x_n, \lambda_n)$ tending to y_0. Since $f_2(x_n, y_n, \lambda_n) \geq 0$ (as $x_n \in S(\lambda_n)$), assumption (ii) gives a contradiction.

If $x_0 \in S(\bar{\lambda}) \subseteq U$, one also obtain another contradiction, since $x_n \notin U$ for all n. Thus, S is usc at $\bar{\lambda}$.

(c) Now let $\{(\lambda_n, x_n)\} \to (\bar{\lambda}, x_0)$ with $x_n \in S(\lambda_n)$ but $x_0 \notin S(\bar{\lambda})$. Then, $f_2(x_0, y_0, \bar{\lambda}) < 0$ for some $y_0 \in Z(x_0, \bar{\lambda})$. By the lower semicontinuity of Z, there exists $y_n \in Z(x_n, \lambda_n)$, $\{y_n\} \to y_0$. Since $x_n \in S(\lambda_n)$,

$$f_2(x_n, y_n, \lambda_n) \geq 0.$$

By the upper semicontinuity of f_2 assumed in (ii), $f_2(x_0, y_0, \bar{\lambda}) \geq 0$, which is impossible. Therefore, S is closed at $\bar{\lambda}$.

The essentialness of all assumptions are now explained by the following examples.

Example 3.1 (upper semicontinuity and compactness in (i) are crucial). Let $X = [0,2]$, $\Lambda = [0, +\infty)$, $\bar{\lambda} = 0$, and

$$K(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (0,1] \cup \{2\}, & \text{if } \lambda \neq 0, \end{cases}$$

$$F(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)),$$

where $f_1(x, y, \lambda) = x(x - y)^2(1 + \lambda)$ and $f_2(x, y, \lambda) = e^{\lambda + xy}\, x(x - y)$.

Clearly, K is lsc at 0 and assumption (ii) holds. Easy calculations yield

$$S_1(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (0,1] \cup \{2\}, & \text{if } \lambda \neq 0, \end{cases}$$

and $Z(x, \lambda) = \{x\}$. Hence, assumption (iii) is satisfied. Direct computations give

$$S(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (0,1] \cup \{2\}, & \text{if } \lambda \neq 0, \end{cases}$$

It is evident that S is neither usc nor closed at $\bar{\lambda}$=0. This is caused by the fact that K is neither upper semicontinuous nor compact-valued at $\bar{\lambda}$.

Example 3.2 (lower semicontinuity in (i) cannot be dispensed). Let $X = [-1,1]$, $\Lambda = \{0\} \cup [2, +\infty)$, $\bar{\lambda} = 0$, and

$$K(\lambda) = \begin{cases} [-1,1], & \text{if } \lambda = 0, \\ [0,1] \cup \{-1\}, & \text{if } \lambda \neq 0, \end{cases}$$

$$F(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)),$$

for $f_1(x, y, \lambda) = (y - x)^2(x + y^2 + \lambda)$ and $f_2(x, y, \lambda) = (\lambda + 1)(x - y)$. We easily get

$$S_1(\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ [0,1] \cup \{-1\}, & \text{if } \lambda \neq 0, \end{cases}$$

It is clear that K is usc and compact valued at 0 and assumption (ii) holds. For each $x \in S_1(\lambda)$, $Z(x, \lambda) = \{x\}$. So, assumption (iii) is fulfilled. But,

$$S(\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ [0,1] \cup \{-1\}, & \text{if } \lambda \neq 0 \end{cases}$$

is neither usc nor closed at 0. The reason is that K is not lsc at 0.

Example 3.3 ((iii) cannot be dropped). Let $X = \mathbb{R}$, $\Lambda = [0,1]$, $K(\lambda) \equiv [0,1]$, $\bar{\lambda} = 0$, and $F(x, y, \lambda) = \left(\lambda(x - y), e^{\lambda y}(y - x)\right)$.

Conditions (i) and (ii) clearly hold. By direct calculations, we have

$$S_1(\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0 \end{cases}$$

$$Z(x, \lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{x\}, & \text{if } \lambda \neq 0, \end{cases}$$

and

$$S(\lambda) = \begin{cases} \{0\}, & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0. \end{cases}$$

Clearly S is neither usc nor closed at $\bar{\lambda} = 0$ and (iii) is violated.

Example 3.4 ((ii) is essential). Let $X = \mathbb{R}$, $\Lambda = [0,1]$, $K(\lambda) \equiv [0,1]$, $\bar{\lambda} = 0$, and $F(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda))$, where

$$f_1(x,y,\lambda)=\begin{cases}x-y, & \text{if } \lambda=0,\\ 0, & \text{if } \lambda\neq 0,\end{cases}\qquad f_2(x,y,\lambda)=\begin{cases}x-y, & \text{if } \lambda=0,\\ \lambda(x-y)^2, & \text{if } \lambda\neq 0.\end{cases}$$

Condition (i) obviously holds. Direct computations give

$$S_1(\lambda)=\begin{cases}\{1\}, & \text{if } \lambda=0,\\ [0,1], & \text{if } \lambda\neq 0,\end{cases}\qquad Z(x,\lambda)=\begin{cases}\{x\}, & \text{if } \lambda=0,\\ [0,1], & \text{if } \lambda\neq 0.\end{cases}$$

Hence, assumption (iii) is fulfilled. Notice that $S(\lambda)\equiv S_1(\lambda)$ is neither usc nor closed at $\bar{\lambda}=0$. Assumption (ii) is violated, since, for $x_n=0, y_n=1, \lambda_n=1/n$, one has $f_1(x_n,y_n,\lambda_n)=0$, $f_2(x_n,y_n,\lambda_n)1/n>0$, but $f_1(0,1,0)=f_2(0,1,0)=-1<0$.

Remark 3.2. Using Lemma 2.1, we imply that the conclusions of Theorem 3.1 are still true, if assumption (ii) is replaced by

(ii') f_i is upper pseudocontinuous in $K(\bar{\lambda})\times K(\bar{\lambda})\times\{\bar{\lambda}\}$ for $i=1,2$.

Theorem 3.3. For (LEP_λ), impose the assumptions of Theorem 3.1, and

(iv) $f_2(\cdot,\cdot,\bar{\lambda})$ *is quasimonotone in* $K(\bar{\lambda})\times K(\bar{\lambda})$;

(v) for each $x\in S(\bar{\lambda})$ *and* $y\in S(\bar{\lambda})\setminus\{x\}$, $f_2(x,y,\bar{\lambda})>0$.

Then, S is continuous at $\bar{\lambda}$.

Proof. (a) The upper semicontinuity S is relied on Theorem 3.1.

(b) Now we prove that S is lsc at $\bar{\lambda}$. Suppose that S is not lsc at $\bar{\lambda}$, i.e., there exist a sequence $\{\lambda_n\}\subseteq\Lambda$ with $\lambda_n\to\bar{\lambda}$ and $x_0\in S(\bar{\lambda})$ such that, for every sequence $x_n\in S(\lambda_n)$, $x_n\nrightarrow x_0$. Combining the upper semicontinuity of K with the compactness of $K(\bar{\lambda})$, we can assume that $x_n\to\bar{x}\in K(\bar{\lambda})$. Using the given techniques in the proof of Theorem 3.1, we also establish that $\bar{x}\in S(\bar{\lambda})$. By the contradiction assumption, we imply that $x_0\neq\bar{x}$. From (v), we have

$$f_2(x_0,\bar{x},\bar{\lambda})>0,$$

and

$$f_2(\bar{x},x_0,\bar{\lambda})>0,$$

which is a contradiction with the quasimonotonicity of $f_2(\cdot,\cdot,\bar{\lambda})$.

We give two examples to show that the assumptions (iv) and (v) in Theorem 3.3 cannot be dropped.

Example 3.5 ((iv) is indispensable). Let $X=\mathbb{R}$, $\Lambda=[0,1]$, $K(\lambda)\equiv[\lambda,\lambda+1]$, $\bar{\lambda}=0$, and $F(x,y,\lambda)=(f_1(x,y,\lambda),f_2(x,y,\lambda))$, where

$$f_1(x,y,\lambda)=0,\quad f_2(x,y,\lambda)=\begin{cases}(x-y)^2(2-y), & \text{if } \lambda=0,\\ (x-y)^2\left(x-\lambda-\frac{1}{2}\right), & \text{if } \lambda\neq 0.\end{cases}$$

It is clear that conditions (i) and (ii) hold. Direct computations give

$$S_1(\lambda)=[\lambda,\lambda+1],\ Z(x,\lambda)=[\lambda,\lambda+1],$$

and

$$S(\lambda)=\begin{cases}[0,1], & \text{if } \lambda=0,\\ \left[\lambda+\frac{1}{2},\lambda+1\right], & \text{if } \lambda\neq 0.\end{cases}$$

Hence, assumptions (iii) and (v) are fulfilled. But, S is not continuous at $\bar{\lambda}=0$. The reason is that assumption (iv) is violated. Indeed, for $x=1, y=0$, and $\bar{\lambda}=0$, one has $f_2(x,y,\bar{\lambda})=2>0$, but $f_2(y,x,\bar{\lambda})=1\nless 0$.

Example 3.6 ((v) is essential). Let $X=\mathbb{R}$, $\Lambda=[0,1]$, $K(\lambda)\equiv[0,\lambda+1]$, $\bar{\lambda}=0$, and $F(x,y,\lambda)=(f_1(x,y,\lambda),f_2(x,y,\lambda))$, where $f_1(x,y,\lambda)=|(x-y)(y-\lambda-1/3)|$, and $f_2(x,y,\lambda)=\lambda(x-y)$.

We have

$$S_1(\lambda)=[0,\lambda+1],\text{and } Z(x,\lambda)=\{x,\lambda+1/3\}.$$

Clearly, assumptions (i)-(iv) are satisfied. Direct computations give

$$S(\lambda)=\begin{cases}[0,1], & \text{if } \lambda=0,\\ \left[\lambda+\frac{1}{3},\lambda+1\right], & \text{if } \lambda\neq 0.\end{cases}$$

and then S is discontinuous at $\bar{\lambda}=0$. The cause is that (v) is not fulfilled.

Remark 3.4. In Anh *et al.,* 2015, by using techniques related to implicit theorems, the authors obtained sufficient conditions for the solution sets of lexicographic equilibrium problems in Banach spaces to be upper semicontinuous and continuous. In this paper, we consider lexicographic equilibrium problem in metric spaces, i.e., linear structures are skipped, Remark 3.2 together with Theorem 3.3 derive Theorems 3.1 and 3.2 in Anh *et al.,* 2015.

4 WELL-POSEDNESS PROPERTIES OF (LEP)

For each number $\varepsilon \in [0;\infty)$, we consider the following approximate problem:

(LEP$_{\lambda,\varepsilon}$) Find $\bar{x} \in K(\lambda)$ such that

$$f_1(\bar{x}, y, \lambda) \geq 0, \quad \forall y \in K(\lambda),$$
$$f_2(\bar{x}, z, \lambda) + \varepsilon \geq 0, \quad \forall z \in Z(\bar{x}, \lambda).$$

This is equivalent to finding $\bar{x} \in S_1(\lambda)$ such that

$$f_2(\bar{x}, z, \lambda) + \varepsilon \geq 0, \forall z \in Z(x, \lambda).$$

The solution set of (LEP$_{\lambda,\varepsilon}$) is denoted by $\tilde{S}(\lambda, \varepsilon)$.

We next recall the notion of well-posedness and continuity-like properties crucial for our analysis in this study.

Definition 4.1. A sequence $\{x_n\}$ with $x_n \in K(\lambda_n)$ is called an *approximating sequence* of (LEP$_{\bar{\lambda}}$) corresponding to a sequence $\{\lambda_n\} \subset \Lambda$ converging to $\bar{\lambda}$ if there is a sequence $\{\varepsilon_n\} \subset (0; +\infty)$ converging to 0 such that $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$ for all n.

Definition 4.2. (**LEP**) is *well-posed* at $\bar{\lambda}$ if for any sequence $\{\lambda_n\}$ in Λ converging to $\bar{\lambda}$, every corresponding approximating sequence of (LEP$_{\bar{\lambda}}$) has a subsequence converging to some point of $S(\bar{\lambda})$.

Definition 4.3. (**LEP**) is *uniquely well-posed* at $\bar{\lambda}$ if:

(i) *$(LEP_{\bar{\lambda}})$ has the unique solution $\bar{x}$,*

(ii) *for any sequence $\{\lambda_n\}$ in Λ converging to $\bar{\lambda}$, every corresponding approximating sequence of $(LEP_{\bar{\lambda}})$ converges to $\bar{x}$.*

Definition 4.4. (**LEP**) is *Hadamard well-posed* at $\bar{\lambda}$ if:

(i) *$(LEP_{\bar{\lambda}})$ has the unique solution $\bar{x}$,*

(ii) for any sequence $\{\lambda_n\}$ in Λ converging to $\bar{\lambda}$, and every sequence $x_n \in S(\lambda_n)$, x_n converges to $\bar{x}$.

Lemma 4.1 (Anh and Khanh, 2011). Let $Q: X \rightrightarrows Y$ be a set-valued mapping between metric spaces. Suppose that $Q(\bar{x})$ is compact. Then, Q is usc at $\bar{x}$ if and only if for any sequence $\{x_n\} \to \bar{x}$, every sequence $\{y_n\}$ with $y_n \in Q(x_n)$ has a subsequence converging to some point in $Q(\bar{x})$. If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.

Theorem 4.2. (**LEP**) is well-posed at $\bar{\lambda}$ if and only if $\tilde{S}$ is upper semicontinuity and has compact valued at $(\bar{\lambda}, 0)$.

Proof. (a) Suppose that for given $\bar{\lambda} \in \Lambda$, $\tilde{S}$ is usc at $(\bar{\lambda}, 0)$ and $\tilde{S}(\bar{\lambda}, 0) = S(\bar{\lambda})$ is compact. Let $\{\lambda_n\} \subseteq \Lambda$ be an arbitrary sequence converging to $\bar{\lambda}$ and $\{x_n\}$ be an approximating sequence for (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$. Then, there exists a sequence $\{\varepsilon_n\} \downarrow 0$ such that, for each $n \in \mathbb{N}$, $x_n \in K(\lambda_n, x_n)$ and

$$f_1(x_n, y, \lambda_n) \geq 0, \quad \forall y \in K(\lambda_n),$$
$$f_2(x_n, z, \lambda_n) + \varepsilon_n \geq 0, \quad \forall z \in Z(x_n, \lambda_n).$$

i.e., $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$. Using the compactness of $\tilde{S}(\bar{\lambda}, 0)$ and the upper semicontinuity of $\tilde{S}$ at $(\bar{\lambda}, 0)$, Lemma 4.1 implies the existence of the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of $S(\bar{\lambda})$. Therefore, (**LEP**) is well-posed at $\bar{\lambda}$.

(b) Conversely, suppose that (**LEP**) is well-posed at $\bar{\lambda}$. Let $\{(\lambda_n, \varepsilon_n)\} \in \Lambda \times \mathbb{R}_+$, with $\{(\lambda_n, \varepsilon_n)\} \to (\bar{\lambda}, 0)$ and $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$. Then, for each $n \in \mathbb{N}$, $x_n \in K(\lambda_n)$, and

$$f_1(x_n, y, \lambda_n) \geq 0, \quad \forall y \in K(\lambda_n),$$
$$f_2(x_n, z, \lambda_n) + \varepsilon_n \geq 0, \quad \forall z \in Z(x_n, \lambda_n).$$

i.e., $\{x_n\}$ is an approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$. Using the well-posedness of (**LEP**), we obtain a subsequence of $\{x_n\}$ converging to some point of $S(\bar{\lambda}) = \tilde{S}(\bar{\lambda}, 0)$. Lemma 4.1 implies the fact that, $\tilde{S}$ is upper semicontinuous and has compact valued at $(\bar{\lambda}, 0)$.

Passing to Hadamard well-posedness for (**LEP**), with the given arguments as in the proof of Theorem 4.2, we establish the following results.

Theorem 4.3. (**LEP**) is Hadamard well-posed at $\bar{\lambda}$ if and only if S is upper semicontinuity and has compact valued at $\bar{\lambda}$.

Theorem 4.4. *Assume that*

(i) *K is continuous at $\bar{\lambda}$ and $K(\bar{\lambda})$ is compact;*

(ii) *f_i is upper semicontinuous in $K(\bar{\lambda}) \times K(\bar{\lambda}) \times \{\bar{\lambda}\}$ for $i = 1,2$;*

(iii) *Z is lsc in $S_1(\bar{\lambda}) \times \{\bar{\lambda}\}$.*

Then, the approximate solution map $\tilde{S}$ is usc and has compact valued at $\bar{\lambda}$.

Proof. Putting $g_1(x, y, \lambda, \varepsilon) = f_1(x, y, \lambda), g_2(x, y, \lambda, \varepsilon) = f_2(x, y, \lambda) + \varepsilon$. By the definition of $\tilde{S}$ and g_i, $i = 1,2$, we imply that, $\bar{x} \in \tilde{S}(\lambda, \varepsilon)$ if and only if

$$g_1(\bar{x}, y, \lambda) \geq 0, \quad \forall y \in K(\lambda),$$
$$g_2(\bar{x}, z, \lambda) + \varepsilon_n \geq 0, \quad \forall z \in Z(\bar{x}, \lambda),$$

where Z is defined by

$$Z(x,\lambda):=\begin{cases}\{z\in K(\lambda)\mid g_1(x,z,\lambda)=0\}, & \text{if } (\lambda,x)\in \text{ graph } S_1\\ X & \text{otherwise,}\end{cases}$$

And $S_1(\lambda)=\{x\in K(\lambda): g_1(x,y,\lambda)\geq 0, \forall y\in K(\lambda)\}$, i.e., $\bar{x}$ is a solution of (LEP_λ) corresponding to g_1 and g_2.

From (ii), we imply that, g_i is upper semicontinuous in $K(\bar{\lambda})\times K(\bar{\lambda})\times\{\bar{\lambda}\}\times\mathbb{R}_+$ for $i=1,2$. Applying Theorem 3.1, we obtain the upper semicontinuity of $\tilde{S}$ and the compactness of $\tilde{S}(\bar{\lambda},0)$.

Combining Remark 3.2, Theorems 4.2, 4.3, 4.4 and Lemma 4.1, we derive the following results.

Corollary 4.5. For (LEP_λ), assume that all assumptions of Theorem 4.2 are satisfied at $\bar{\lambda}$. Then (**LEP**) is well-posed at $\bar{\lambda}$. In addition, if $S(\bar{\lambda})$ is singleton, then (**LEP**) is uniquely well-posed at $\bar{\lambda}$.

Corollary 4.6. *For* (LEP_λ), *assume that*

(i) K is continuous at $\bar{\lambda}$ and $K(\bar{\lambda})$ is compact;

(ii) f_i is upper pseudocontinuous in $K(\bar{\lambda})\times K(\bar{\lambda})\times\{\bar{\lambda}\}$ for $i=1,2$;

(iii) Z is lsc in $S_1(\bar{\lambda})\times\{\bar{\lambda}\}$.

Then, (**LEP**) is Hadamard well-posed at $\bar{\lambda}$ if $S(\bar{\lambda})$ is singleton.

5 CONCLUDING REMARKS

In this paper, we consider lexicographic vector equilibrium problems in metric spaces and study stability and well-posedness properties of solution mappings. Since equilibrium problems contain as special cases many optimization-related models such as variational inequalities, constrained minimization, fixed-point and coincidence point problems, complementarity problems, minimax problems, etc., consequences for these problems can be deduced from our results. We have illustrated the essentialness of assumptions imposed in our results by examples. Extensions of the stability and well-posedness in parametric lexicographic vector equilibrium problems with constraints depending on the state variable (known also as lexicographic vector quasiequilibrium problems) may be a possible development of this paper.

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