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The hit problem of three variables for the cohomology of the classifying space BE_s as a module over the mod 3 Steenrod Algebra at some low degrees

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ABSTRACT

The hit problem, set up by F. Peterson, finds a minimal set of generators for the polynomial algebra $P(s) = \mathbb{F}_2[x_1, x_2, \dots, x_s]$, *as a module over the mod-2 Steenrod algebra. We study the extended hit problem for the cohomology of the classifying space* BE_s *over field* $\mathbb{F}_3$, $P(s) = H^*B(\mathbb{R}P^\infty)^s = E(x_1, x_2, \dots, x_s) \otimes \mathbb{F}_p[y_1, y_2, \dots, y_s]$, *with* $s = 3$ *at degrees* $d \leq 10$.

1. INTRODUCTION

Let p be a prime number and E_s be the s-dimension $\mathbb{F}_p$-vector space. The cohomology of the classifying space BE_s over the field $\mathbb{F}_p$ is defined by

$$P(s) = H^*BE_s = E(x_1, x_2, \dots, x_s) \otimes \mathbb{F}_p[y_1, y_2, \dots, y_s],$$

as a graded left module over the mod p Steenrod algebra $\mathcal{A}_p$. The mod p Steenrod algebra $\mathcal{A}_p$ acts by the composition of linear operators on $P(s)$ and the action of the Steenrod power $\mathcal{P}^i$, $(i \geq 0)$ and the Bockstein operation β is determined by the Cartan formula and its elementary properties (Steenrod, 1962).

Finding a minimal set of generators of $P(s)$ was initiated by Peterson (1987). This problem is called the hit problem. $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$. The quotient of the left $\mathcal{A}_p$-module $P(s)$ by the hit elements in $\mathcal{A}^+ P(s)$ is denoted by $QP(s) = P(s)/\mathcal{A}^+ P(s) = \mathbb{F}_p \otimes_{\mathcal{A}_p} P(s)$, which is the set of all elements f, called hit element, in $P(s)$ represented in the form

$$f = \sum_{i \geq 0} \gamma_i \beta^{\varepsilon_1} \mathcal{P}^i \beta^{\varepsilon_2}(f_i),$$

where $f_i \in P^{d-[2(p-1)i+\varepsilon_1+\varepsilon_2]}(s), \gamma_i \in \mathbb{F}_p$, $\varepsilon_1, \varepsilon_2 \in \{0,1\}$. In other words, we want to find a basis of the $\mathbb{F}_p$-vector space $QP(s)$.

Immediately after being suggested by Peterson (1987, 1989), the hit problem was studied by numerous authors such as Wood (1989); Singer (1989); and Priddy (1990). The vector space $QP(s)$ over field $\mathbb{F}_2$ was explicitly calculated by Peterson (1987) for $s = 1, 2$, by Kameko (1990) for $s = 3$. The case $s = 4$ has been treated by Kameko (2003) and Sum (2007). Sum (2014) also explicitly calculated the hit problem of 5 variables at some generic degrees.

Many respectable results on the hit problem have been studied by numerous authors over the field $\mathbb{F}_2$ for over three decades (Singer, 1989, 1991; Wood, 1989, 1992, 2000; Boardman, 1993; Hung & Peterson, 1995, 1998; Minami, 1999; Giambalvo & Peterson, 2001; Hung and Nam, 2001a, 2001b; Janfada & Wood, 2002; Bruner et al., 2005; Hung, 2005; Ha, 2007; Sum, 2007, 2013, 2014, 2023; Mothebe, 2013; Phuc, 2020, 2022, 2023; Tin, 2022a, 2022b). Meanwhile, little is known about the hit problem on the field $\mathbb{F}_p$, where p is an odd prime. With an odd prime p, Crossley (1996, 1999) calculated the dimensions of $QP(s)$ at some generic degrees for the case $s = 1$, and $s = 2$. In this paper, the author determines the hit elements of $P(3)$ with $s = 3$ at degrees $d \leq 10$.

The paper is organized as follows. Section 2 provides preliminary results of the hit problem and Steenrod algebra. We calculate in Section 3 the actions of the admissible basis of the Steenrod algebra $\mathcal{A}_3$ on the elements in $P(3)$ used to check whether or not these elements are hit. Finally, Section 4 discusses the achieved results and provides some open problems for further research.

2. PRELIMINARIES

In this section, we present some basic results about Steenrod algebra studied by Steenrod (1962) and Minami (1999) for the case of an odd prime p with more specific results for the case $p = 3$.

2.1. Steenrod algebra over the field $\mathbb{F}_3$

Let $\mathbb{F}_3$ be a field including 3 elements $\{0,1,2\}$ and X be a topological space over the field $\mathbb{F}_3$. For all integers $i \geq 0$ and $n \geq 0$, the Steenrod power is a homomorphism defined by

$$\mathcal{P}^i : H^n(X, \mathbb{F}_3) \to H^{n+4i}(X, \mathbb{F}_3).$$

The homomorphism $\beta : H^n(X, \mathbb{Z}_3) \to H^{n+1}(X, \mathbb{Z}_3)$ is called the Bock-stein coboundary operator associated with the short exact coefficient sequence

$$0 \to \mathbb{Z}_3 \to \mathbb{Z}_{3^2} \to \mathbb{Z}_3 \to 0.$$

We define the mod 3 Steenrod algebra $\mathcal{A}_3$ to be the graded associative algebra generated by the elements $\mathcal{P}^i$ of degree $4i$ and β of degree 1, subject to β^2=0, $\mathcal{P}^0 = 1$, and the Adem relations

$$\mathcal{P}^i\mathcal{P}^j = \sum_{t=0}^{[i/3]} (-1)^{i+t} \binom{2(j-t)-1}{i-3t} \mathcal{P}^{i+j-t}\mathcal{P}^t,$$

for $i < 3j$, and

$$\mathcal{P}^i\beta\mathcal{P}^j = \sum_{t=0}^{[i/3]} (-1)^{i+t} \binom{2(j-t)}{i-3t} \beta\mathcal{P}^{i+j-t}\mathcal{P}^t$$
$$- \sum_{t=0}^{[(i-1)/3]} (-1)^{i-1+t} \binom{2(j-t)-1}{i-3t-1} \mathcal{P}^{i+j-t}\beta\mathcal{P}^t,$$

for $i \leq 3j$.

Let $(\mathbb{R}P^\infty)^3$ be a 3-dimensional $\mathbb{F}_3$-infinite real projective space. It is well-known that the mod 3 cohomology of the classifying space $B(\mathbb{R}P^\infty)^3$ is given by

$$P(3) = H^*B(\mathbb{R}P^\infty)^3 =$$
$$E(x_1, x_2, x_3) \otimes \mathbb{F}_3[y_1, y_2, y_3],$$

where $E(x_1, x_2, x_3)$ is a notation for the exterior algebra over $\mathbb{F}_3$ generated by variables x_1, x_2, x_3 for degrees 1 and $\mathbb{F}_3[y_1, y_2, y_3]$ is a notation for the polynomial algebra over $\mathbb{F}_3$ generated by variables y_1, y_2, y_3 for degrees 2.

Then,

$$P(3) = Sp\{x_1^{\varepsilon_1} y_1^{i_1} x_2^{\varepsilon_2} y_2^{i_2} x_3^{\varepsilon_3} y_3^{i_3}, \varepsilon_j \in \{0,1\}, i_j \geq 0\},$$

is a module over the mod 3 Steenrod algebra $\mathcal{A}_3$. The action of $\mathcal{A}_3$ on $P(3)$ is explicitly given by

$$\mathcal{P}^i(y_j) = \begin{cases} y_j, & i = 0 \\ y_j^3, & i = 1. \\ 0, & i > 1 \end{cases}$$

The Cartan formula is

$$\mathcal{P}^k(y_i y_j) = \sum_{t=0}^{k} \mathcal{P}^{k-t}(y_i)\mathcal{P}^t(y_j),$$

and $\beta(x_i y_j) = \beta(x_i) y_j + (-1)^{|x_i|} x_i \beta(y_j)$.

Moreover,

1) $\mathcal{P}^i(x_j) = \begin{cases} x_j, & i = 0 \\ 0, & i > 0 \end{cases}$'

2) $\mathcal{P}^i(y_j^k) = \binom{k}{i} y_j^{k+2i}$,

3) $\mathcal{P}^i\left(y_j^{3^k}\right) = \begin{cases} y_j^{3^k}, & i = 0 \\ y_j^{3^{k+1}}, & i = 3^k \\ 0, & i \neq 0 \text{ or } 3^k \end{cases}$,

4) $\beta(x_i) = y_i$, and $\beta(y_j) = 0$.

The admissible basis of degrees $d \leq 10$ of the mod 3 Steenrod algebra $\mathcal{A}_3$: $\mathcal{P}^0$ (degree 0); β (degree 1); $\mathcal{P}^1$ (degree 4); $\beta\mathcal{P}^1$, $\mathcal{P}^1\beta$ (degree 5); $\beta\mathcal{P}^1\beta$ (degree 6); $\mathcal{P}^2$ (degree 8); $\beta\mathcal{P}^2$, $\mathcal{P}^2\beta$ (degree 9); $\beta\mathcal{P}^2\beta$ (degree 10).

2.2. The hit problem

Definition 2.1 Let $P^d(3)$ be the vector space of homogeneous polynomials of degree d

$$P^d(3) = Sp\left\{\prod_{j=1}^{3} x_j^{\varepsilon_j} y_j^{i_j}, \varepsilon_j \in \{0,1\}, \sum_{j=1}^{3}(\varepsilon_j + 2i_j) = d\right\}.$$

Then, $P^d(3)$ is a subspace of $P(3)$. Since $P(3)$ is graded by integers $d \geq 0$, therefore

$$P(3) = \sum_{d\geq 0} P^d(3).$$

Denote by $QP^d(3)$ the subspace of $QP(3)$ comprising all the classes represented by the elements in $P^d(3)$.

Definition 2.2 A homogeneous polynomial $f \in P^d(3)$ in $\mathcal{A}$-module $P(s)$ is hit if it satisfies a hit equation

$$f = \sum_{i>0} \gamma_i \beta^{\varepsilon_1} \mathcal{P}^i \beta^{\varepsilon_2}(f_i), \quad (1)$$

where the homogeneous elements f_i in $P(s)$ have gradings strictly less than $d - [2(p-1)i + \varepsilon_1 + \varepsilon_2]$, denoted by $f_i \in P^{d-[4i+\varepsilon_1+\varepsilon_2]}(3)$, $\gamma_i \in \mathbb{F}_3$, $\varepsilon_j \in \{0,1\}$.

Remark. The decomposition of f in (1) is not unique.

3. RESULTS AND DISCUSSIONS

3.1. The action of an admissible monomials of degrees $d \leq 10$ in $\mathcal{A}_3$ on P_3

For all $f \in P(3)$, f has the form

$$f = x_1^{\varepsilon_1} y_1^{i_1} x_2^{\varepsilon_2} y_2^{i_2} x_3^{\varepsilon_3} y_3^{i_3}, \ \varepsilon_j \in \{0,1\}, i_j \geq 0.$$

Since all the elements in the admissible basis of $\mathcal{A}_3$ have the form $\beta^{\epsilon_1}\mathcal{P}^{i_1}\beta^{\epsilon_2}\mathcal{P}^{i_2}\cdots\beta^{\epsilon_s}\mathcal{P}^{i_s}$, where $\epsilon_j \in \{0,1\}$, $i_j \geq 3i_{j+1} + \epsilon_j$. We only need to calculate the action of β and $\mathcal{P}^i$ on the elements f of $P(3)$.

By direct calculation, we have the following results

$$\beta(f) = \beta(x_1^{\varepsilon_1}) y_1^{i_1} x_2^{\varepsilon_2} y_2^{i_2} x_3^{\varepsilon_3} y_3^{i_3} + (-1)^{\varepsilon_1} x_1^{\varepsilon_1} y_1^{i_1} \beta(x_2^{\varepsilon_2}) y_2^{i_2} x_3^{\varepsilon_3} y_3^{i_3} + (-1)^{\varepsilon_1+\varepsilon_2} x_1^{\varepsilon_1} y_1^{i_1} x_2^{\varepsilon_2} y_2^{i_2} \beta(x_3^{\varepsilon_3}) y_3^{i_3},$$

where $\beta\left(x_j^{\varepsilon_j}\right) = \begin{cases} y_j, & \varepsilon_j = 1 \\ 0, & \varepsilon_j = 0 \end{cases}, j = 1,2,3.$

$$\mathcal{P}^i(f) = \sum_{k_1+k_2+k_3=i} \alpha_i x_1^{\varepsilon_1} y_1^{i_1+2k_1} x_2^{\varepsilon_2} y_2^{i_2+2k_2} x_3^{\varepsilon_3} y_3^{i_3+2k_3},$$

where $\alpha_i = \binom{i_1}{k_1}\binom{i_2}{k_2}\binom{i_3}{k_3} \bmod 3$.

3.2. Hit elements of $P(3)$ at degrees $d \leq 10$

For all $f \in P^d(3)$, f has the form

$$f = x_1^{\varepsilon_1} y_1^{i_1} x_2^{\varepsilon_2} y_2^{i_2} x_3^{\varepsilon_3} y_3^{i_3},$$

where $\varepsilon_j \in \{0,1\}, i_j \geq 0, \sum_{j=1}^{3}(\varepsilon_j + 2i_j) = d$.

Because $\deg x_i = 1$ and $\deg y_i = 2$, we have that

if d is odd, $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ can only be a permutation of (1,0,0) or (1,1,1), or if d is even, $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ can only be a permutation of (0,0,0) or (1,1,0),

and (i_1, i_2, i_3) can only be a permutation of triples that satisfy $i_1 + i_2 + i_3 = [d - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)]/2$.

Then, the number of generators of $P^d(3)$ is calculated as in table 1

Table 1. The number of elements of $P^d(3)$

Degree d	Number of elements in $P^d(3)$	Degree d	Number of elements in $P^d(3)$
1	3	6	28
2	6	7	36
3	10	8	45
4	15	9	55
5	21	10	66

We consider the following cases.

3.2.1. The case $d = 1$

$$P^1(3) = Sp\{x_1, x_2, x_3\}.$$

All three elements x_1, x_2, x_3 are not hit in $P(3)$.

So $\dim QP^1(3) = 3$.

3.2.2. The case $d = 2$

$$P^2(3) = Sp\{x_1x_2, x_1x_3, x_2x_3, y_1, y_2, y_3\}.$$

We get $y_i = \beta(x_i)$ so y_i is hit, and x_ix_j is not hit in $P(3)$. So $\dim QP^2(3) = 3$.

3.2.3. The case $d = 3$

The $P^3(3)$ is generated by elements of the following form

1) x_iy_j for $1 \leq i \leq j \leq 3$,

2) y_ix_j for $1 \leq i < j \leq 3$,

3)$x_1x_2x_3$.

Because $\beta(x_i x_j) = y_i x_j + 2x_i y_j$, $(1 \le i < j \le 3)$, and $\beta(y_i) = 0$, $(1 \le i \le 3)$, all elements in $P(3)$ are not hit. So $\dim QP^3(3) = 10$.

3.2.4. The case $d = 4$

The $P^4(3)$ is generated by elements of the following form

1) y_i^2, $1 \le i \le 3$,
2) $y_i y_j$, $1 \le i < j \le 3$,
3) $x_i y_j x_k$, $1 \le i \le j < k \le 3$,
4) $x_i x_j y_k$, $1 \le i < j \le k \le 3$,
5) $y_1 x_2 x_3$.

Since

$$y_i^2 = \beta(x_i y_i), y_i y_j = \beta(x_i y_j),$$

$$\beta(x_1 x_2 x_3) = y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3,$$

so elements y_i^2, $y_i y_j$ are hit and elements $x_i y_j x_k$, $x_i x_j y_k$, $y_i x_j x_k$ are not hit in $P(3)$. Then, $\dim QP^4(3) = 9$.

3.2.5. The case $d = 5$

The $P^5(3)$ is generated by elements of the following form

1) $x_i y_j^2$, $1 \le i \le j \le 3$,
2) $y_i^2 x_j$, $1 \le i < j \le 3$,
3) $x_i y_j y_k$, $1 \le i \le j < k \le 3$,
4) $y_i x_j y_k$, $1 \le i < j \le k \le 3$,
5) $y_1 y_2 x_3$,
6) $x_1 y_1 x_2 x_3$, $x_1 x_2 y_2 x_3$, $x_1 x_2 x_3 y_3$.

For $f \in P^5(3)$, f is hit if and only if f can be expressed as the sum of $\beta(g)$ and $\mathcal{P}^1(h)$, where $g \in P^4(3)$, $h \in P^1(3)$. When we act on $\mathcal{P}^1$ and β respectively on the elements of $P^1(3)$ and $P^4(3)$), we have

$$\mathcal{P}^1(x_j) = 0,$$

$$\beta(y_i^2) = \beta(y_i y_j) = 0,$$

$$\beta(x_i y_j x_k) = y_i y_j x_k + 2x_i y_j y_k,$$

$$\beta(x_i x_j y_k) = y_i x_j y_k + 2x_i y_j y_k,$$

$$\beta(y_i x_j x_k) = y_i y_j x_k + 2y_i x_j y_k,$$

All element of $P^5(3)$ are not hit in $P(3)$.

So $\dim QP^5(3) = 21$

3.2.6. The case $d = 6$

The $P^6(3)$ is generated by elements of the following form

1) y_i^3, $1 \le i \le 3$,
2) $y_i^2 y_j$, $y_i y_j^2$, $1 \le i < j \le 3$,
3) $y_1 y_2 y_3$,
4) $x_i x_j y_k^2$, $1 \le i < j \le k \le 3$,
5) $x_i y_j^2 x_k$, $1 \le i \le j < k \le 3$,
6) $y_1^2 x_2 x_3$,
7) $x_i y_j x_k y_l$, $1 \le i \le j < k \le l \le 3$,
8) $x_1 x_2 y_2 y_3$, $x_1 y_1 y_2 x_3$, $y_1 x_2 y_2 x_3$, $y_1 x_2 x_3 y_3$.

For $f \in P^6(3)$, f is hit if and only if f can be expressed as the sum of $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, and $\beta\mathcal{P}^1(f_4)$, where $f_1 \in P^5(3)$; $f_2 \in P^2(3)$; f_3, $f_4 \in P^1(3)$.

We get

$$y_i^3 = \mathcal{P}^1(y_i), y_i y_j^2 = \beta(x_i y_j^2), y_i^2 y_j = \beta(y_i^2 x_j),$$

$$y_1 y_2 y_3 = \beta(y_1 y_2 x_3),$$

$$\beta(x_1 y_1 x_2 x_3) = y_1^2 x_2 x_3 + 2x_1 y_1 y_2 x_3 + x_1 y_1 x_2 y_3,$$

$$\beta(x_1 x_2 y_2 x_3) = y_1 x_2 y_2 x_3 + x_1 y_2^2 x_3 + 2x_1 x_2 y_2 y_3,$$

$$\beta(x_1 x_2 x_3 y_3) = y_1 x_2 x_3 y_3 + x_1 y_2 x_3 y_3 + 2x_1 x_2 y_3^2,$$

$$\mathcal{P}^1(x_i x_j) = 0, \beta\mathcal{P}^1(x_i) = 0, \mathcal{P}^1\beta(x_i) = y_i^3.$$

Therefore, y_i^3, $y_i y_j^2$, $y_i^2 y_j$, $y_1 y_2 y_3$ are hit in $P(3)$, others are not hit. So $\dim QP^6(3) = 18$.

3.2.7. The case $d = 7$

The $P^7(3)$ is generated by elements of the following form

1) $x_i y_j^3$, $1 \le i \le j \le 3$,
2) $y_i^3 x_j$, $1 \le i < j \le 3$,
3) $x_i y_j^2 y_k$, $x_i y_j y_k^2$, $1 \le i \le j < k \le 3$,
4) $y_i^2 x_j y_k$, $y_i x_j y_k^2$, $1 \le i < j \le k \le 3$,
5) $y_1^2 y_2 x_3$, $y_1 y_2^2 x_3$,
6) $x_1 y_1 y_2 y_3$, $y_1 x_2 y_2 y_3$, $y_1 y_2 x_3 y_3$,
7) $x_1 y_1^2 x_2 x_3$, $x_1 x_2 y_2^2 x_3$, $x_1 x_2 x_3 y_3^2$,
8) $x_1 y_1 x_2 y_2 x_3$, $x_1 x_2 y_2 x_3 y_3$, $x_1 y_1 x_2 x_3 y_3$.

For $f \in P^7(3)$, f is hit if and only if f can be expressed as the sum of $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, $\beta\mathcal{P}^1(f_4)$, and $\beta\mathcal{P}^1\beta(f_5)$ where $f_1 \in P^6(3)$, $f_2 \in P^3(3)$, f_3, $f_4 \in P^2(3)$, $f_5 \in P^1(3)$.

The elements $x_i y_j^3$, $y_i^3 x_j$ are hit as

$$x_iy_j^3 = \mathcal{P}^1(x_iy_j), y_i^3x_j = \mathcal{P}^1(y_ix_j).$$

Moreover, we get

$$\beta\mathcal{P}^1\beta(x_i) = 0,$$
$$\beta\mathcal{P}^1(x_ix_j) = \beta\mathcal{P}^1(y_i) = \mathcal{P}^1\beta(y_i) = 0,$$
$$\mathcal{P}^1\beta(x_ix_j) = y_i^3x_j + 2x_iy_j^3,$$
$$\beta\mathcal{P}^1\beta(x_i) = 0,\ \mathcal{P}^1(x_1x_2x_3) = 0,$$
$$\beta(y_i^3) = \beta(y_i^2y_j) = \beta(y_iy_j^2) = \beta(y_iy_jy_k) = 0,$$
$$\beta(x_ix_jy_k^2) = y_ix_jy_k^2 + 2x_iy_jy_k^2,$$
$$\beta(x_iy_j^2x_k) = y_iy_j^2x_k + 2x_iy_j^2y_k,$$
$$\beta(y_1^2x_2x_3) = y_1^2y_2x_3 + 2y_1^2x_2y_3,$$
$$\beta(x_iy_jx_ky_l) = y_iy_jx_ky_l + 2x_iy_jy_ky_l,$$
$$\beta(x_1x_2y_2y_3) = y_1x_2y_2y_3 + 2x_1y_2^2y_3,$$
$$\beta(x_1y_1y_2x_3) = y_1^2y_2x_3 + 2x_1y_1y_2y_3,$$
$$\beta(y_1x_2y_2x_3) = y_1y_2^2x_3 + 2y_1x_2y_2y_3,$$
$$\beta(y_1x_2x_3y_3) = y_1y_2x_3y_3 + 2y_1x_2y_3^2.$$

The aforementioned calculations show that the other elements are not hit. So $\dim QP^7(3) = 27$.

3.2.8. The case $d = 8$

The generators of $P^8(3)$ have the following form

1) $y_i^4, 1 \le i \le 3,$
2) $y_i^3y_j, y_iy_j^3, y_i^2y_j^2, 1 \le i < j \le 3,$
3) $y_1^2y_2y_3, y_1y_2^2y_3, y_1y_2y_3^2,$
4) $x_iy_j^3x_k, 1 \le i \le j < k \le 3,$
5) $x_ix_jy_k^3,\ 1 \le i < j \le k \le 3,$
6) $y_1^3x_2x_3,$
7) $x_iy_j^2x_ky_l, x_iy_jx_ky_l^2, 1 \le i \le j < k \le l \le 3,$
8) $x_1y_1^2y_2x_3, x_1y_1y_2^2x_3, x_1x_2y_2^2y_3, x_1x_2y_2y_3^2,$ $y_1x_2y_2^2x_3, y_1x_2x_3y_3^2, y_1^2x_2y_2x_3, y_1^2x_2x_3y_3,$
9) $x_1y_1x_2y_2y_3, y_1x_2y_2x_3y_3, x_1y_1y_2x_3y_3.$

We get

$$y_i^4 = 2\mathcal{P}^1(y_i^2),$$
$$y_iy_j^3 = \beta(x_iy_j^3) = \beta\mathcal{P}^1(x_iy_j),$$
$$y_i^3y_j = \beta(y_i^3x_j) = \beta\mathcal{P}^1(y_ix_j),$$
$$y_i^2y_j^2 = \beta(y_i^2x_jy_j),$$
$$y_1^2y_2y_3 = \beta(y_1^2x_2y_3) = \beta(x_1y_1y_2y_3),$$
$$y_1y_2^2y_3 = \beta(x_1y_2^2y_3) = \beta(y_1x_2y_2y_3),$$
$$y_1y_2y_3^2 = \beta(x_1y_2y_3^2) = \beta(y_1y_2x_3y_3),$$
$$x_ix_jy_k^3 = \mathcal{P}^1(x_ix_jy_k), x_iy_j^3x_k = \mathcal{P}^1(x_iy_jx_k),$$
$$y_1^3x_2x_3 = \mathcal{P}^1(y_1x_2x_3).$$

That shows these elements are hit.

For $f \in P^8(3)$, f is hit if and only if f can be expressed as the sum of $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, $\beta\mathcal{P}^1(f_4)$, and $\beta\mathcal{P}^1\beta(f_5)$, where $f_1 \in P^7(3); f_2 \in P^4(3);\ f_3, f_4 \in P^3(3);\ f_5 \in P^2(3)$. The others are not hit because

$$\beta\mathcal{P}^1\beta(x_ix_j) = y_i^3y_j + 2y_iy_j^3,\ \beta\mathcal{P}^1\beta(y_i) = 0,$$
$$\mathcal{P}^1\beta(x_iy_j) = \mathcal{P}^1\beta(y_ix_j) = y_i^3y_j + 2y_iy_j^3,$$
$$\mathcal{P}^1\beta(x_1x_2x_3) = y_1^3x_2x_3 + x_1y_2^3x_3 + x_1x_2y_3^3,$$
$$\mathcal{P}^1(y_iy_j) = y_i^3y_j + y_iy_j^3,$$
$$\beta(x_1y_1^2x_2x_3) = y_1^3x_2x_3 + 2x_1y_1^2y_2x_3 + x_1y_1^2x_2y_3,$$
$$\beta(x_1x_2y_2^2x_3) = y_1x_2y_2^2x_3 + 2x_1y_2^3x_3 + x_1x_2y_2^2y_3,$$
$$\beta(x_1x_2x_3y_3^2) = y_1x_2x_3y_3^2 + 2x_1y_2x_3y_3^2 + x_1x_2y_3^3,$$
$$\beta(x_1y_1x_2y_2x_3) = y_1^2x_2y_2x_3 + 2x_1y_1y_2^2x_3 + x_1y_1x_2y_2y_3,$$
$$\beta(x_1x_2y_2x_3y_3) = y_1x_2y_2x_3y_3 + 2x_1y_2^2x_3y_3 + x_1x_2y_2y_3^2,$$
$$\beta(x_1y_1x_2x_3y_3) = y_1^2x_2x_3y_3 + 2x_1y_1y_2x_3y_3 + x_1y_1x_2y_3^2.$$

So $\dim QP^8(3) = 21$.

3.2.9. The case $d = 9$

The generators of $P^9(3)$ have the following form

1) $x_iy_j^4, 1 \le i \le j \le 3,$
2) $y_i^4x_j, 1 \le i < j \le 3,$
3) $x_iy_j^3y_k, x_iy_jy_k^3, x_iy_j^2y_k^2, 1 \le i \le j < k \le 3,$
4) $y_i^3x_jy_k, y_ix_jy_k^3, y_i^2x_jy_k^2, 1 \le i < j \le k \le 3,$
5) $y_1^3y_2x_3, y_1y_2^3x_3, y_1^2y_2^2x_3,$
6) $x_1y_1y_2y_3^2, y_1x_2y_2y_3^2, y_1y_2x_3y_3^2, x_1y_1y_2^2y_3,$ $y_1x_2y_2^2y_3, y_1y_2^2x_3y_3, x_1y_1^2y_2y_3, y_1^2x_2y_2y_3,$ $y_1^2y_2x_3y_3, x_1y_1^3x_2x_3, x_1x_2y_2^3x_3, x_1x_2x_3y_3^3,$ $x_1y_1^2x_2y_2x_3, x_1x_2y_2^2x_3y_3, x_1y_1x_2x_3y_3^2,$ $x_1y_1x_2y_2^2x_3, x_1x_2y_2x_3y_3^2, x_1y_1^2x_2x_3y_3,$ $x_1y_1x_2y_2x_3y_3.$

For $f \in P^9(3)$, f is hit if and only if f can be expressed as the sum of $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, $\beta\mathcal{P}^1(f_4)$, $\beta\mathcal{P}^1\beta(f_5)$, $\mathcal{P}^2(f_6)$, $\mathcal{P}^2\beta(f_7)$, $\beta\mathcal{P}^2(f_8)$, where $f_1 \in P^8(3)$, $f_2 \in P^5(3)$, f_3, $f_4 \in P^4(3)$, $f_5 \in P^3(3)$, $f_6 \in P^1(3)$.

We get

$$y_i^4x_j = 2\mathcal{P}^1(y_i^2x_j),\ x_iy_j^4 = 2\mathcal{P}^1(x_iy_j^2),$$

$$\mathcal{P}^1(x_1y_1x_2x_3) = x_1y_1^3x_2x_3,$$

$$\mathcal{P}^1(x_1x_2y_2x_3) = x_1x_2y_2^3x_3,$$

$$\mathcal{P}^1(x_1x_2x_3y_3) = x_1x_2x_3y_3^3,$$

$$y_1y_2^3x_3 = \mathcal{P}^1\beta(x_1x_2y_3) + 2\beta\mathcal{P}^1(x_1x_2y_3) + \beta\mathcal{P}^1(y_1x_2x_3) + 2\beta\mathcal{P}^1(x_1y_2x_3) + 2\mathcal{P}^1(y_1y_2x_3),$$

$$y_1^3y_2x_3 = 2\mathcal{P}^1\beta(x_1x_2y_3) + \beta\mathcal{P}^1(x_1x_2y_3) + 2\beta\mathcal{P}^1(y_1x_2x_3) + \beta\mathcal{P}^1(x_1y_2x_3) + 2\mathcal{P}^1(y_1y_2x_3),$$

$$x_iy_jy_k^3 = \mathcal{P}^1\beta(y_ix_jx_k) + 2\beta\mathcal{P}^1(y_ix_jx_k) + \beta\mathcal{P}^1(x_ix_jy_k) + 2\beta\mathcal{P}^1(x_iy_jx_k) + 2\mathcal{P}^1(x_iy_jy_k),$$

$$x_iy_j^3y_k = 2\mathcal{P}^1\beta(y_ix_jx_k) + \beta\mathcal{P}^1(y_ix_jx_k) + 2\beta\mathcal{P}^1(x_ix_jy_k) + \beta\mathcal{P}^1(x_iy_jx_k) + 2\mathcal{P}^1(x_iy_jy_k),$$

$$y_i^3x_jy_k = 2\mathcal{P}^1\beta(x_iy_jx_k) + \beta\mathcal{P}^1(y_ix_jx_k) + \beta\mathcal{P}^1(x_ix_jy_k) + \beta\mathcal{P}^1(x_iy_jx_k) + 2\mathcal{P}^1(x_iy_jy_k),$$

$$y_ix_jy_k^3 = \mathcal{P}^1\beta(x_iy_jx_k) + 2\beta\mathcal{P}^1(y_ix_jx_k) + 2\beta\mathcal{P}^1(x_ix_jy_k) + 2\beta\mathcal{P}^1(x_iy_jx_k) + 2\mathcal{P}^1(x_iy_jy_k).$$

This proves that elements $x_iy_j^4, y_i^4x_j, x_iy_j^3y_k, y_i^3x_jy_k, y_1^3y_2x_3, x_iy_jy_k^3,$ $y_ix_jy_k^3,$

$y_1y_2^3x_3, x_1y_1^3x_2x_3, x_1x_2y_2^3x_3, x_1x_2x_3y_3^3$ are hit.

The action of the admissible basis of $\mathcal{A}$ on the elements of $P^d(3)$, with degrees $d = 1, 3, 4, 5, 8$, is calculated as follows.

With $f_1 \in P^8(3)$, if f_1 contains no elements x_i then $\beta(f_1) = 0$, and

$$\beta(x_1y_2^2x_3y_3) = y_1y_2^2x_3y_3 + 2x_1y_2^2y_3^2,$$

$$\beta(x_1y_1^2y_2x_3) = y_1^3y_2x_3 + 2x_1y_1^2y_2y_3,$$

$$\beta(x_1y_1y_2^2x_3) = y_1^2y_2^2x_3 + 2x_1y_1y_2^2y_3,$$

$$\beta(x_1x_2y_2^2y_3) = y_1x_2y_2^2y_3 + 2x_1y_2^3y_3,$$

$$\beta(x_1x_2y_2y_3^2) = y_1x_2y_2y_3^2 + 2x_1y_2^2y_3^2,$$

$$\beta(y_1x_2y_2^2x_3) = y_1y_2^3x_3 + 2y_1x_2y_2^2y_3,$$

$$\beta(y_1x_2x_3y_3^2) = y_1y_2x_3y_3^2 + 2y_1x_2y_3^3,$$

$$\beta(y_1^2x_2y_2x_3) = y_1^2y_2^2x_3 + 2y_1^2x_2y_2y_3,$$

$$\beta(y_1^2x_2x_3y_3) = y_1^2y_2x_3y_3 + 2y_1^2x_2y_3^2,$$

$$\beta(x_1y_1x_2y_3^2) = y_1^2x_2y_3^2 + 2x_1y_1y_2y_3^2,$$

$$\beta(x_iy_jx_ky_l^2) = y_iy_jx_ky_l^2 + 2x_iy_jy_ky_l^2,$$

$$\beta(x_1y_1x_2y_2y_3) = y_1^2x_2y_2y_3 + 2x_1y_1y_2^2y_3,$$

$$\beta(y_1x_2y_2x_3y_3) = y_1y_2^2x_3y_3 + 2y_1x_2y_2y_3^2,$$

$$\beta(x_1y_1y_2x_3y_3) = y_1^2y_2x_3y_3 + 2x_1\,y_1y_2y_3^2.$$

With $f_2 \in P^5(3)$, we get

$$\mathcal{P}^1(x_iy_jy_k) = x_iy_j^3y_k + x_iy_jy_k^3,$$

$$\mathcal{P}^1(y_ix_jy_k) = y_i^3x_jy_k + y_ix_jy_k^3,$$

$$\mathcal{P}^1(y_iy_jx_k) = y_i^3y_jx_k + y_iy_j^3x_k.$$

With $f_3, f_4 \in P^4(3)$, we get

$$\mathcal{P}^1\beta(y_i^2) = \mathcal{P}^1\beta(y_iy_j) = 0,$$

$$\mathcal{P}^1\beta(x_ix_jy_k) = y_i^3x_jy_k + y_ix_jy_k^3 + 2x_iy_j^3y_k + 2x_iy_jy_k^3,$$

$$\mathcal{P}^1\beta(x_iy_jx_k) = y_i^3y_jx_k + y_iy_j^3x_k + 2x_iy_j^3y_k + 2x_iy_jy_k^3,$$

$$\mathcal{P}^1\beta(y_ix_jx_k) = y_i^3y_jx_k + y_iy_j^3x_k + 2y_i^3x_jy_k + 2y_ix_jy_k^3,$$

$$\beta\mathcal{P}^1(y_i^2) = \beta\mathcal{P}^1(y_iy_j) = 0,$$

$$\beta\mathcal{P}^1(x_ix_jy_k) = y_ix_jy_k^3 + 2x_iy_jy_k^3,$$

$$\beta\mathcal{P}^1(x_iy_jx_k) = y_iy_j^3x_k + 2x_iy_j^3y_k,$$

$$\beta\mathcal{P}^1(y_ix_jx_k) = y_i^3y_jx_k + 2y_i^3x_jy_k.$$

With $f_5 \in P^3(3)$, we get

$$\beta\mathcal{P}^1\beta(x_iy_j) = \beta\mathcal{P}^1\beta(y_ix_j) = 0,$$

$$\beta\mathcal{P}^1\beta(x_1x_2x_3) = y_i^3y_jx_k + 2y_i^3x_jy_k + 2y_iy_j^3x_k + x_iy_j^3y_k + y_ix_jy_k^3 + 2x_iy_jy_k^3.$$

With $f_6 \in P^1(3)$, we get $\mathcal{P}^2(x_i) = 0$.

The above results show that the elements $x_1y_1^2x_2y_2x_3$, $x_1x_2y_2^2x_3y_3$, $x_1y_1x_2x_3y_3^2$, $x_1y_1x_2y_2^2x_3$, $x_1x_2y_2x_3y_3^2$, $x_1y_1^2x_2x_3y_3$, $x_1y_1x_2y_2x_3y_3$ do not appear in the action of admissible monomials in $\mathcal{A}$ on $P(3)$. Hence, these elements cannot be expressed as the sum of $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, $\beta\mathcal{P}^1(f_4)$,

$\beta\mathcal{P}^1\beta(f_5), \mathcal{P}^2(f_6), \mathcal{P}^2\beta(f_7), \beta\mathcal{P}^2(f_8)$ that mean they are not hit.

Moreover, elements containing y_i^2 appear only in $\beta(f_1),\ f_1 \in P^8(3)$ but

$$\beta(y_1x_2y_2x_3y_3) = \beta(x_1y_2^2x_3y_3) + 2\beta(x_1x_2y_2y_3^2),$$

$$\beta(x_1y_1x_2y_2y_3) = \beta(x_1y_1y_2^2x_3) + 2\beta(y_1^2x_2y_2x_3),$$

$$\beta(x_1y_1y_2x_3y_3) = \beta(y_1^2x_2x_3y_3) + \beta(x_1y_1x_2y_3^2),$$

$$\beta\mathcal{P}^1\beta(x_1x_2x_3) = \beta\mathcal{P}^1(y_ix_jx_k) + 2\beta\mathcal{P}^1(x_iy_jx_k) + \beta\mathcal{P}^1(x_ix_jy_k).$$

So $\dim QP^9(3) = 25$.

3.2.10. The case $d = 10$

The $P^{10}(3)$ is generated by elements of the following form

1) $y_1^{i_1}y_2^{i_2}y_3^{i_3},\ i_1, i_2, i_3 \geq 0$ and $i_1 + i_2 + i_3 = 5$,
2) $x_iy_j^4x_k, x_ix_jy_k^4,\ 1 \leq i \leq j \leq k \leq 3$,
3) $y_1^4x_2x_3$,
4) $x_iy_j^3x_ky_l,\ x_iy_jx_ky_l^3,\ x_iy_j^2x_ky_l^2,\ 1 \leq i \leq j < k \leq l \leq 3$,
5) $x_1y_1^3y_2x_3,\ x_1y_1y_2^3x_3,\ x_1x_2y_2^3y_3,\ x_1x_2y_2y_3^3, y_1x_2y_2^3x_3, y_1^3x_2y_2x_3, y_1x_2x_3y_3^3, y_1^3x_2x_3y_3$,
6) $x_1y_1^2y_2^2x_3, x_1x_2y_2^2y_3^2, y_1^2x_2y_2^2x_3, y_1^2x_2x_3y_3^2$,
7) $x_1y_1^2x_2y_2y_3, x_1y_1x_2y_2^2y_3, x_1y_1x_2y_2y_3^2$,

$y_1^2x_2y_2x_3y_3,\quad y_1x_2y_2^2x_3y_3,\quad y_1x_2y_2x_3y_3^2, x_1y_1^2y_2x_3y_3$,

$x_1y_1y_2^2x_3y_3, x_1y_1y_2x_3y_3^2$.

Direct computations show that elements of forms 1, 2, and 3 are both hit.

The form 1: (i_1, i_2, i_3) includes cases and their permutations, respectively (5,0,0), (4,1,0), (3,2,0), (2,2,1), (1,1,3)

$$y_i^5 = \beta(x_iy_i^4), y_i^4y_j = \beta(y_i^4x_j) = 2\beta\mathcal{P}^1(y_i^2x_j),$$

$$y_iy_j^4 = \beta(x_iy_j^4) = 2\beta\mathcal{P}^1(x_iy_j^2),$$

$$y_i^3y_j^2 = \beta(x_iy_i^2y_j^2), y_i^2y_j^3 = \beta(y_i^2y_j^2x_j),$$

$$y_1^2y_2^2y_3 = \beta(y_1^2y_2^2x_3) = \beta(x_1y_1y_2^2y_3) = \beta(y_1^2x_2y_2y_3),$$

$$y_1^2y_2y_3^2 = \beta(y_1^2x_2y_3^2) = \beta(x_1y_1y_2y_3^2) = \beta(y_1^2y_2x_3y_3),$$

$$y_1y_2^2y_3^2 = \beta(x_1y_2^2y_3^2) = \beta(y_1x_2y_2y_3^2) = \beta(y_1y_2^2x_3y_3),$$

$$y_1y_2y_3^3 = \beta(x_1y_2y_3^3) = \beta(y_1y_2x_3y_3^2),$$

$$y_1y_2^3y_3 = \beta(x_1y_2^3y_3) = \beta(y_1x_2y_2^2y_3),$$

$$y_1^3y_2y_3 = \beta(y_1^3y_2x_3) = \beta(x_1y_1^2y_2y_3).$$

The form 2 & 3:

$$x_iy_j^4x_k = 2\mathcal{P}^1(x_iy_j^2x_k),$$

$$y_1^4x_2x_3 = 2\mathcal{P}^1(y_1^2x_2x_3),$$

$$x_ix_jy_k^4 = 2\mathcal{P}^1(x_ix_jy_k^2).$$

For $f \in P^{10}(3)$, f is hit if and only if f can be expressed as the sum of $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, $\beta\mathcal{P}^1(f_4)$, $\beta\mathcal{P}^1\beta(f_5)$, $\mathcal{P}^2(f_6)$, $\mathcal{P}^2\beta(f_7)$, $\beta\mathcal{P}^2(f_8)$ where $f_1 \in P^9(3)$; $f_2 \in P^6(3)$; $f_3, f_4 \in P^5(3)$; $f_5 \in P^4(3)$; $f_6 \in P^2(3)$; $f_7, f_8 \in P^1(3)$.

The other actions of $\mathcal{A}$ on $P^{10}(3)$

$$\beta(x_1y_1^3x_2x_3) = y_1^4x_2x_3 + 2x_1y_1^3y_2x_3 + x_1y_1^3x_2y_3,$$

$$\beta(x_1x_2y_2^3x_3) = y_1x_2y_2^3x_3 + 2x_1y_2^4x_3 + x_1x_2y_2^3y_3,$$

$$\beta(x_1x_2x_3y_3^3) = y_1x_2x_3y_3^3 + 2x_1y_2x_3y_3^3 + x_1x_2y_3^4,$$

$$\beta(x_1y_1^2x_2y_2x_3) = y_1^3x_2y_2x_3 + 2x_1y_1^2y_2^2x_3 + x_1y_1^2x_2y_2y_3,$$

$$\beta(x_1x_2y_2^2x_3y_3) = y_1x_2y_2^2x_3y_3 + 2x_1y_2^3x_3y_3 + x_1x_2y_2^2y_3^2,$$

$$\beta(x_1y_1x_2x_3y_3^2) = y_1^2x_2x_3y_3^2 + 2x_1y_1y_2x_3y_3^2 + x_1y_1x_2y_3^3,$$

$$\beta(x_1y_1x_2y_2^2x_3) = y_1^2x_2y_2^2x_3 + 2x_1y_1y_2^3x_3 + x_1y_1x_2y_2^2y_3,$$

$$\beta(x_1x_2y_2x_3y_3^2) = y_1x_2y_2x_3y_3^2 + 2x_1y_2^2x_3y_3^2 + x_1x_2y_2y_3^3,$$

$$\beta(x_1y_1^2x_2x_3y_3) = y_1^3x_2x_3y_3 + 2x_1y_1^2y_2x_3y_3 + x_1y_1^2x_2y_3^2,$$

$$\beta(x_1y_1x_2y_2x_3y_3) = y_1^2x_2y_2x_3y_3 + x_1y_1y_2^2x_3y_3 + x_1y_1x_2y_2y_3^2,$$

$$\mathcal{P}^1(y_i^3) = 0,$$

$$\mathcal{P}^1(y_i^2y_j) = 2y_i^4y_j + y_i^2y_j^3,$$

$$\mathcal{P}^1(y_iy_j^2) = y_i^3y_j^2 + 2y_iy_j^4,$$

$$\mathcal{P}^1(y_iy_jy_k) = y_i^3y_jy_k + y_iy_j^3y_k + y_iy_jy_k^3,$$

$$\mathcal{P}^1(x_iy_jx_ky_l) = x_iy_j^3x_ky_l + x_iy_jx_ky_l^3,$$

$$\mathcal{P}^1(x_1x_2y_2y_3) = x_1x_2y_2^3y_3 + x_1x_2y_2y_3^3,$$

$$\mathcal{P}^1(x_1y_1y_2x_3) = x_1y_1^3y_2x_3 + x_1y_1y_2^3x_3,$$

$$\mathcal{P}^1(y_1x_2y_2x_3) = y_1^3x_2y_2x_3 + y_1x_2y_2^3x_3,$$
$$\mathcal{P}^1(y_1x_2x_3y_3) = y_1^3x_2x_3y_3 + y_1x_2x_3y_3^3,$$
$$\mathcal{P}^1\beta(x_iy_j^2) = y_i^3y_j^2 + 2y_iy_j^4,$$
$$\mathcal{P}^1\beta(y_i^2x_j) = 2y_i^4y_j + y_i^2y_j^3,$$
$$\mathcal{P}^1\beta(x_iy_jy_k) = \mathcal{P}^1\beta(y_iy_jx_k) = \mathcal{P}^1\beta(y_ix_jy_k)$$
$$= y_i^3y_jy_k + y_iy_j^3y_k + y_iy_jy_k^3,$$
$$\mathcal{P}^1\beta(x_1y_1x_2x_3) = 2y_1^4x_2x_3 + 2x_1y_1^3y_2x_3$$
$$+2x_1y_1y_2^3x_3 + x_1y_1^3x_2y_3 + x_1y_1x_2y_3^3,$$
$$\mathcal{P}^1\beta(x_1x_2y_2x_3) = y_1^3x_2y_2x_3 + y_1x_2y_2^3x_3$$
$$+2x_1y_2^4x_3 + 2x_1x_2y_2^3y_3 + 2x_1x_2y_2y_3^3,$$
$$\mathcal{P}^1\beta(x_1x_2x_3y_3) = y_1^3x_2x_3y_3 + y_1x_2x_3y_3^3$$
$$+2x_1y_2^3x_3y_3 + 2x_1y_2x_3y_3^3 + 2x_1x_2y_3^4,$$
$$\beta\mathcal{P}^1(x_iy_jy_k) = y_iy_j^3y_k + y_iy_jy_k^3,$$
$$\beta\mathcal{P}^1(y_iy_jx_k) = y_i^3y_jy_k + y_iy_j^3y_k,$$
$$\beta\mathcal{P}^1(y_ix_jy_k) = y_i^3y_jy_k + y_iy_jy_k^3,$$
$$\beta\mathcal{P}^1(x_1y_1x_2x_3) = \beta(x_1y_1^3x_2x_3),$$
$$\beta\mathcal{P}^1(x_1x_2y_2x_3) = \beta(x_1x_2y_2^3x_3),$$
$$\beta\mathcal{P}^1(x_1x_2x_3y_3) = \beta(x_1x_2x_3y_3^3),$$
$$\beta\mathcal{P}^1\beta(y_i^2) = 0, \beta\mathcal{P}^1\beta(y_iy_j) = 0,$$
$$\beta\mathcal{P}^1\beta(x_ix_jy_k) = y_i^3y_jy_k + 2y_iy_j^3y_k,$$
$$\beta\mathcal{P}^1\beta(x_iy_jx_k) = y_i^3y_jy_k + 2y_iy_jy_k^3,$$
$$\beta\mathcal{P}^1\beta(y_ix_jx_k) = y_iy_j^3y_k + 2y_iy_jy_k^3,$$
$$\mathcal{P}^2(x_ix_j) = \mathcal{P}^2(y_i) = 0,$$
$$\beta\mathcal{P}^2(x_i) = \mathcal{P}^2\beta(x_i) = 0.$$

The form 4 & 5: From the above calculation results, we see that the elements in the form 4 & 5 cannot be the sum of the elements $\beta(f_1)$, $\mathcal{P}^1(f_2)$, $\mathcal{P}^1\beta(f_3)$, $\beta\mathcal{P}^1(f_4)$, $\mathcal{P}^1\beta(f_5)$, $\mathcal{P}^2(f_6)$, $\mathcal{P}^2\beta(f_7)$, $\beta\mathcal{P}^2(f_8)$, where $f_1 \in P^9(3)$; $f_2 \in P^6(3)$; $f_3, f_4 \in P^5(3)$; $f_5 \in P^4(3)$; $f_6 \in P^2(3)$; $f_7, f_8 \in P^1(3)$. In other words, these elements are not hit.

For example:

Element $x_iy_j^2x_ky_l^2$ appears only in $\beta(x_1x_2y_2x_3y_3^2)$ or $\beta(x_1y_1^2x_2x_3y_3)$ corresponding to different (i,j,k,l) tuples $(1 \le i \le j < k \le l \le 3)$.

Element $x_1y_1^2y_2^2x_3$ appears only in $\beta(x_1y_1^2x_2y_2x_3)$.

Element $x_1x_2y_2^2y_3^2$ appears only in $\beta(x_1x_2y_2^2x_3y_3)$.

Element $y_1^2x_2y_2^2x_3$ appears only in $\beta(x_1y_1x_2y_2^2x_3)$.

Element $y_1^2x_2x_3y_3^2$ appears only in $\beta(x_1y_1x_2x_3y_3^2)$.

That proves these elements are not hit.

The form 6 & 7: In view of the above calculation results, we see that these elements in the form 6 & 7 appear exactly once in the effects of β on the elements $x_1y_1^2x_2y_2x_3$, $x_1x_2y_2^2x_3y_3$, $x_1y_1x_2x_3y_3^2$, $x_1y_1x_2y_2^2x_3$, $x_1x_2y_2x_3y_3^2$, $x_1y_1^2x_2x_3y_3$, $x_1y_1x_2y_2x_3y_3$. This means that these elements are not hit. So $\dim QP^{10}(3) = 36$.

4. CONCLUSION

Determining the hit element by direct computation is a quite complicated approach, so this study is only for explicitly calculating a few small cases d to visualize the early picture of $QP^d(3)$. For general cases, other efficient tools are needed to reduce the computational costs. This is for the author's research directions in the future.

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